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TECHNICAL NOTES

Fully developed combined heat and mass transfer natural convection between parallel plates with asymmetric boundary conditions

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INTRODUCTION

IN THIS note, closed form solutions for hydrodynamically and thermally fully developed laminar natural convection in a parallel plate channel are presented. The fluid motion is generated by the combined buoyancy effects due to temperature and concentration differences between fluid in the channel and fluid surrounding the channel. The two buoyancy mechanisms may aid or oppose each other and the fluid may rise or fall in the channel.

Previous studies of free convection combined heat and mass transfer have been for external flows. Gebhart and Pera [1] obtained similarity solutions for vertical flat plate and plume flows and gave a summary of past studies. Chen and Yuh [2] studied inclined flat plate flows that also allow similarity solutions.

Fully developed natural convection flow in a parallel wall channel has been treated for heat transfer only by Bodoia and Osterle [3] for uniform, equal temperature walls and by Ostrach [4], Aung [5], Miyatake and Fujii [6–8] and Miyatake *et al.* [9] for asymmetric heating. Gill *et al.* [10] considered combined free and forced convection for fully developed flow between inclined parallel plates with asymmetric mass transfer.

The boundary conditions considered in this study are uniform wall temperature and concentration (UWT/C) and uniform heat and mass fluxes (UH/MF) with the opposing wall maintained at some fraction between 0 and 1 of the corresponding value on the primary reference wall. The finite length channel is considered to be deep relative to the width and the fluid is constrained to enter and exit the channel at the ambient hydrostatic pressure prevailing at that elevation.

The geometry and boundary conditions are shown in Fig. 1. When the channel length l is much longer than the width b, the flow approaches a fully developed condition in the majority of the channel. The limits for which the fully developed assumption is valid are determined in refs. [11, 12] from numerical results for the developing flow. The following analysis of combined heat and mass transfer parallels that of Aung [5] for the heat transfer only case.

ANALYSIS AND RESULTS

For small temperature and concentration differences, the surface normal velocity may be neglected and the usual Boussinesq approximations and equation of state may be employed (see Gebhart and Pera [1]). Under the fully developed assumption that $U = U(Y)$, the non-dimensional governing equations for the steady, two-dimensional laminar flow of a constant property, Newtonian fluid with negligible viscous dissipation, thermal diffusion and diffusion-thermo reduce to

$$\frac{dP}{dX}-\frac{d^2U}{dY^2} = \theta(X,Y)+NC(X,Y) \tag{1}$$

$$Pr\,U\frac{\partial}{\partial X}\theta(X,Y) = \frac{\partial^2}{\partial Y^2}\theta(X,Y) \tag{2}$$

$$Sc\,U\frac{\partial}{\partial X}C(X,Y) = \frac{\partial^2}{\partial Y^2}C(X,Y) \tag{3}$$

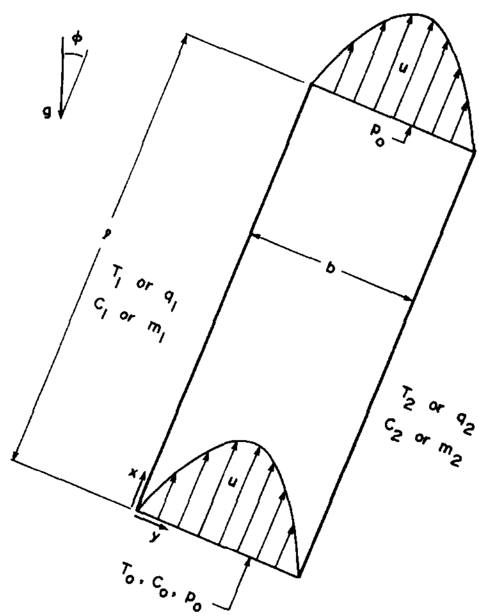

FIG. 1. Geometry and boundary conditions.

NOMENCLATURE

b	plate spacing [m]
C	species mass fraction
D	species diffusivity [$m^2 s^{-1}$]
g	gravitational acceleration [$m s^{-2}$]
Gr	channel Grashof number based on b and l
k	thermal conductivity [$W m^{-1} K^{-1}$]
l	channel length [m]
L	non-dimensional channel length
Le	Lewis number, $Sc/Pr = \alpha/D$
m	mass flux [$kg s^{-1} m^{-2}$]
N	buoyancy due to mass transfer parameter
Nu	Nusselt number based on dimension b
p	pressure [Pa]
p'	local hydrostatic pressure [Pa]
P	dimensionless pressure
Pr	Prandtl number, ν/α
q	heat flux [$W m^{-2}$]
Q	dimensionless volume flow rate per unit depth
Ra	Rayleigh number, $Gr\,Pr$
Sc	Schmidt number, ν/D
Sh	Sherwood number based on dimension b
T	fluid temperature [K]
u	velocity in the x-direction [$m s^{-1}$]
x	distance along the channel [m]
y	distance across the channel [m].

Greek symbols

α	thermal diffusivity [$m^2 s^{-1}$]
β	volumetric coefficient of thermal expansion
β^*	volumetric coefficient of concentration expansion
θ	dimensionless fluid temperature
ν	kinematic viscosity [$m^2 s^{-1}$]
ρ	fluid density [$kg m^{-3}$]
ϕ	angle of inclination from vertical.

Subscripts

0	channel inlet
1	reference wall at $y = 0$
2	wall at $y = b$
c	concentration
h	heat flux
L	channel exit
m	mass flux
t	temperature.

where

$$X = \frac{x}{l\,Gr}, \quad Y = \frac{y}{b}$$

$$U = \frac{b^2 u}{l\nu\,Gr}, \quad P = \frac{(p-p')b^4}{\rho l^2 \nu^2 Gr^2}.$$

For uniform wall temperature and concentration (UWT/C)

$$\theta = \frac{T-T_0}{T_1-T_0}, \qquad C = \frac{C-C_0}{C_1-C_0}$$

$$Gr = \frac{g\cos\phi\,\beta(T_1-T_0)b^4}{l\nu^2}, \quad N = \frac{\beta^*(C_1-C_0)}{\beta(T_1-T_0)}.$$

For uniform heat and mass fluxes (UH/MF)

$$\theta = \frac{T-T_0}{q_1 b/k}, \qquad C = \frac{C-C_0}{m_1 b/D}$$

$$Gr = \frac{g\cos\phi\,\beta q_1 b^5}{l\nu^2 k}, \quad N = \frac{\beta^* m_1/D}{\beta q_1/k}.$$

In the above, subscript 0 refers to the inlet condition and subscript 1 refers to the hotter wall at $Y = 0$. Using subscript 2 to refer to the wall at $Y = 1$, the wall temperature and concentration difference ratios and heat and mass flux ratios may be defined as

$$r_t = \frac{T_2-T_0}{T_1-T_0}, \quad r_c = \frac{C_2-C_0}{C_1-C_0}$$

$$r_h = \frac{q_2}{q_1}, \qquad r_m = \frac{m_2}{m_1}.$$

By forming $\partial/\partial X$ of equation (1), substituting equations (2) and (3) and then forming $\partial^2/\partial Y^2$ of equation (1), equations (1)–(3) can be combined into a single equation only if the Lewis number is unity ($Sc = Pr$). A solution $U = U(Y)$ is possible only if d^2P/dX^2 is a constant independent of X, say γ. Applying the conditions $P = 0$ at $X = 0$ and L (where L is the value of X at $x = l$, i.e. $L = 1/Gr$), there results from direct integration

$$P(X) = \frac{\gamma X}{2}(X-L) \tag{4}$$

where γ is an eigenvalue to be determined. Equations (1)–(3) are then

$$\frac{d^4}{dY^4}U(Y)+\lambda^4 U(Y) = 0 \tag{5}$$

$$\gamma\left(X-\frac{L}{2}\right)-\frac{d^2U}{dY^2}(Y) = \theta(X,Y)+NC(X,Y) \tag{6}$$

where

$$\lambda^4 = \gamma\,Pr.$$

Uniform wall temperature/concentration

For UWT/C, the boundary conditions are

$$U(0) = U(1) = 0 \tag{7a}$$

$$\theta(Y=0) = 1, \quad \theta(Y=1) = r_t \tag{7b}$$

$$C(Y=0) = 1, \quad C(Y=1) = r_c. \tag{7c}$$

For fully developed flow, the wall shear is constant. Since the inlet (rather than local bulk) values have been chosen in the definition of θ and C and the UWT/C boundary conditions are constant with X, the temperature, concentration, and buoyancy force are all invariant with X. Thus, for UWT/C in fully developed flow, the pressure is constant and $\gamma = 0$. Hence, $\lambda = 0$. The boundary conditions (7b) and (7c) may be added together to yield appropriate boundary conditions for equation (6). The solution of equation (5) is then

$$U(Y) = (1+N)\left\{\frac{(1-R)}{6}(Y^3-Y)-\frac{1}{2}(Y^2-Y)\right\} \tag{8}$$

where

$$R = \frac{r_t+Nr_c}{1+N}. \tag{9}$$

Note that the velocity does not depend on the Prandtl number. The volume flow rate is

$$Q = \int_0^1 U\,dY = (1+N)\frac{(1+R)}{24}. \tag{10}$$

To insure a positive flow rate, $N \geqslant -1$ and $R > -1$; if $N = -1$, then $r_t > r_c$. A larger r_t or N gives rise to higher velocities and flow rates. For aiding buoyancy due to mass transfer (N positive), a larger r_c generates a larger flow rate while for opposing buoyancy (N negative) a larger r_c reduces the flow rate. This induced flow rate in dimensionless form is in all cases independent of the dimensionless channel length or Grashof number.

The solution of equations (2), (3) and (7) yields

$$\theta = 1-(1-r_t)Y \quad (11)$$

$$C = 1-(1-r_c)Y. \quad (12)$$

The total heat added to the fluid is

$$H = \int_0^1 U\theta \,\mathrm{d}Y = (1+N)\left\{\frac{7(r_t+R)+8(1+r_tR)}{360}\right\}. \quad (13)$$

An average Nusselt number for both walls of the channel may be obtained from an energy balance as

$$\overline{Nu_b} = \tfrac{1}{2}H\,Pr\,Gr. \quad (14)$$

The local Nusselt number has the same magnitude and opposite sign for each wall and may be obtained with the aid of equation (11) as

$$Nu_{b,1} = -\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = (1-r_t) \quad (15a)$$

$$Nu_{b,2} = \left.\frac{\partial\theta}{\partial Y}\right|_{Y=1} = -(1-r_t). \quad (15b)$$

For fully developed flow with symmetric wall temperatures ($r_t = 1$), the local Nusselt number is zero. All of the energy that can be transferred to the fluid has been transferred ($\theta = 1$ for all Y). For asymmetric wall temperatures, heat is conducted from the hot wall through the fluid to the cool wall.

The total species added to the fluid, M, is

$$M = \int_0^1 UC\,\mathrm{d}Y = (1+N)\left\{\frac{7(r_c+R)+8(1+r_cR)}{360}\right\}. \quad (16)$$

The average Sherwood number for both walls of the channel is

$$\overline{Sh_b} = \tfrac{1}{2}M\,Sc\,Gr. \quad (17)$$

As for the local Nusselt number, the local Sherwood number has the same magnitude and opposite sign for each wall

$$Sh_{b,1} = -\left.\frac{\partial C}{\partial Y}\right|_{Y=0} = (1-r_c) \quad (18a)$$

$$Sh_{b,2} = \left.\frac{\partial C}{\partial Y}\right|_{Y=1} = -(1-r_c). \quad (18b)$$

Temperature and concentration are independent of Prandtl and Schmidt number, respectively. This situation can be explained by observing that all the diffusion of heat and mass that could take place has taken place. Thus, the earlier restriction that $Le = 1$ may be relaxed for the UWT/C case.

It is interesting to note that the symmetric UWT/C solution ($r_t = r_c = 1$) has zero heat and mass fluxes at both walls. Thus, this solution is also valid for the case where wall 1 is held at a uniform temperature and concentration and wall 2 has zero heat and mass fluxes. In this instance wall 2 will attain T_1 and C_1 naturally by diffusion of heat and mass across the channel.

Uniform heat/mass flux

For UH/MF, the boundary conditions are

$$U(0) = U(1) = 0 \quad (19a)$$

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = -1, \quad \left.\frac{\partial\theta}{\partial Y}\right|_{Y=1} = r_h \quad (19b)$$

$$\left.\frac{\partial C}{\partial Y}\right|_{Y=0} = -1, \quad \left.\frac{\partial C}{\partial Y}\right|_{Y=1} = r_m. \quad (19c)$$

A solution to equations (5), (6) and (19) can be obtained only for $Le = 1$ and $r_h = r_m$. Under these conditions, the analytical solution is very similar to the heat transfer only UHF case of Aung [5]. An approximate solution method is given below so that more general conditions can be considered.

Inspection of the analytical solution for UH/MF reveals that the velocity profile is very close to a symmetric parabola, at least for L sufficiently large, independent of the θ and C boundary conditions. Thus, it can be shown that

$$U = 6QY(1-Y) \quad (20)$$

which satisfies the boundary conditions. The parabolic profile can be used to study the effects of Lewis number different from unity and $r_h \neq r_m$. For a Lewis number much smaller than unity and asymmetric boundary conditions, one would expect an asymmetric velocity profile for large N, but the parabolic profile still provides the limiting behavior for a sufficiently small Rayleigh number (large L).

For UH/MF boundary conditions, the temperature field must vary linearly with X. An energy balance equating the heat added to the fluid up to X to the heat conducted from the walls may be formulated. Using the velocity profile of equation (20), the solution for the temperature profile is then

$$\theta(X, Y) = (1+r_h)\left\{\frac{X}{Pr\,Q} - \frac{Y^4}{2} + Y^3 - \frac{9}{70}\right\} - Y + \frac{1}{2}. \quad (21)$$

In an analogous manner, the solution for the concentration profile is obtained as

$$C(X, Y) = (1+r_m)\left\{\frac{X}{Sc\,Q} - \frac{Y^4}{2} + Y^3 - \frac{9}{70}\right\} - Y + \frac{1}{2}. \quad (22)$$

The maximum wall temperature and concentration occur at the channel exit and vary inversely with the flow rate Q.

To relate L and Q, a momentum balance from the channel inlet to exit may be used. No net pressure force acts on the fluid and the fluid gains no momentum due to the fully developed assumption. The balance reduces to buoyancy overcoming wall shear. The result is

$$\frac{Q}{Z} = \left[\frac{1}{12Z\,Ra}\right]^{1/2} \quad (23)$$

where

$$Z = [1+r_h+N(1+r_m)/Le]/2. \quad (24)$$

This result is identical to that obtained from the analytical solution if $Le = 1$. For $Le < 1$ and N positive, a higher flow rate is obtained relative to $Le = 1$.

The local Nusselt number can be found from the inverse of the temperature evaluated at the wall. For wall 1 at $Y = 0$

$$Nu_{b,1} = \frac{q_1 b}{k(T_1-T_0)} = \left[(1+r_h)\left\{\frac{x}{l}\left[\frac{12}{Z\,Ra}\right]^{1/2} - \frac{9}{70}\right\} + \frac{1}{2}\right]^{-1}. \quad (25)$$

Similarly, for wall 2 at $Y = 1$

$$Nu_{b,2} = \frac{r_h q_1 b}{k(T_1-T_0)} = r_h\left[(1+r_h)\left\{\frac{x}{l}\left[\frac{12}{Z\,Ra}\right]^{1/2} + \frac{13}{35}\right\} - \frac{1}{2}\right]^{-1}. \quad (26)$$

Since the average wall temperature is equal to the local wall temperature at $x = l/2$ for small Ra, the average or midheight Nusselt number for each wall can be found by evaluating equations (25) and (26) at $x/l = 1/2$. The local Sherwood number for wall 1 is

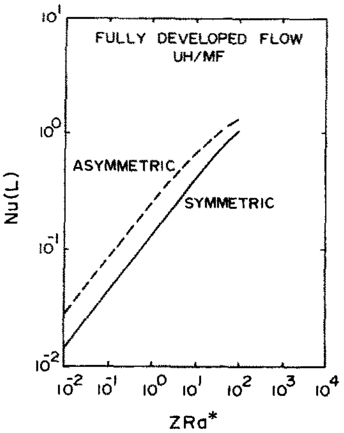

FIG. 2. Approximate solution exit Nusselt number.

$$Sh_{b,1} = \frac{m_1 b}{D(C_1 - C_0)} = \left[(1+r_m)\left\{ \frac{x}{l\,Le}\left[\frac{12}{Z\,Ra}\right]^{1/2} - \frac{9}{70} \right\} + \frac{1}{2} \right]^{-1}. \quad (27)$$

For wall 2

$$Sh_{b,2} = \frac{r_m m_1 b}{D(C_1 - C_0)} = r_m \left[(1+r_m)\left\{ \frac{x}{l\,Le}\left[\frac{12}{Z\,Ra}\right]^{1/2} + \frac{13}{35} \right\} - \frac{1}{2} \right]^{-1}. \quad (28)$$

The average Sherwood number for each wall can be found by evaluating equations (27) and (28) at $x/l = 1/2$.

The local and average Nu and Sh are seen to increase with Ra and N. For positive N, increasing Le increases Sh while decreasing Nu. A similar behavior was found for external UH/MF flows by Chen and Yuh [2]. Increasing r_h decreases Nu and increases Sh while the reverse holds for r_m. In particular, note that $Nu_{b,1}$ and $Sh_{b,1}$ for $r_h = r_m = 0$ are greater than $Nu_{b,1}$ $(=Nu_{b,2})$ and $Sh_{b,1}$ $(=Sh_{b,2})$ for $r_h = r_m = 1$ at equal Ra. This behavior is opposite of that found for forced convection duct flows where the highest transfer parameters are for symmetric boundary conditions [13].

In the approximate solution method, the energy balance used requires the bulk temperature to be zero at $X = 0$ which is not imposed in the analytical solution. This gives rise to the constant terms in equations (21), (25) and (26). These terms are only significant for $Z\,Ra > 1$ as shown in Fig. 2. The approximate fully developed solution for $Nu(L)$ is accurate to within 5% for $Z\,Ra$ as high as 10 and shows the proper trend for developing flow at even higher $Z\,Ra$ [11]. In comparison, Aung [5] found that the analytical heat transfer solution was accurate only up to $Ra = 0.14$.

CONCLUSIONS

An analytical solution for combined heat and mass transfer natural convection between parallel plates has been presented for the fully developed flow case. For UWT/C, the velocity, temperature and concentration are independent of Pr, Sc (or Le) and Gr. A larger wall temperature ratio (r_t) or buoyancy due to mass transfer (N) induces a larger volume flow rate. For positive N, a larger wall concentration ratio (r_c) produces a larger flow rate while the converse is true if $-1 < N < 0$.

For UH/MF, an analytical solution can be obtained only for $Le = 1$. Further, the temperature and concentration (and thus Nu and Sh) can be found only if they are equal (i.e. $r_h = r_m$). These requirements limit the usefulness of the analytical solution for UH/MF.

An approximate solution for UH/MF is obtained for general Le and boundary conditions by assuming a parabolic velocity profile. The flow rate for UH/MF depends on the boundary conditions and N in a manner similar to UWT/C. For UH/MF, however, the dimensionless flow rate decreases in proportion to the square root of Ra and increases for decreasing Le. The local Nu and Sh depend on Ra, N and Le in a manner similar to external flows. The highest Nu and Sh (or lowest wall temperature and concentration) are found for asymmetric boundary conditions which are opposite of the results for forced convection duct flows.

Practical buoyancy-driven duct flows are rarely fully developed over the entire duct length. However, the present results are useful in understanding and correlating results for developing flows.

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